6.2 Vector Space and Inner Product Space in $\mathbb{R}^n$

In linear algebra, an inner product space is a vector space[14] with an additional structure called an inner product.

Definition 6.15. When we have a list of vectors, we use **superscripts** in parentheses as indices of vectors. **Subscripts** represent element indices inside individual vectors.

Example 6.16. Here is a list of four vectors:

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

For the second vector, we have $v_1^{(2)} = 1$, $v_2^{(2)} = -1$, and $v_3^{(2)} = 0$.

Definition 6.17. The **inner product** of two real-valued n-dimensional (column) vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T \mathbf{u} = \sum_{k=1}^{n} u_k v_k.$$

In elementary linear algebra class, you may encounter this quantity in the form of the **dot product** between two vectors.

Definition 6.18. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

More generally, a set of N vectors $\mathbf{v}^{(k)}$, $1 \le k \le N$, are **orthogonal** if $\left\langle \mathbf{v}^{(i)}, \mathbf{v}^{(j)} \right\rangle = 0$ for all $1 \le i, j \le N$, and $i \ne j$.

Definition 6.19. The **norm** of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$ and is defined as

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

which in the n-dimensional Euclidean space is simply the **length** of the vector.

Definition 6.20. A collection of vectors is said to be **orthonormal** if the vectors are orthogonal and each vector has a unit norm.

[14] Recall that a vector space is a mathematical structure formed by a collection of elements called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context.

6.21. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can **decompose** $\mathbf{v}$ into a sum of two vectors, one a multiple of $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.

(a) $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$ is the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$.

(b) $\mathbf{v} - \operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ is the component of $\mathbf{v}$ orthogonal to $\mathbf{u}$.

Example 6.22. Let $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

6.23. Suppose $\mathbf{e}$ is a unit vector; that is $\|\mathbf{e}\| = 1$.

$$\operatorname{proj}_{\mathbf{e}}(\mathbf{v}) =$$

6.24. Any vector in a vector space may also be represented as a linear combination of orthogonal unit vectors or an **orthonormal basis** $\left\{\mathbf{e}^{(i)}, 1 \le i \le N\right\}$ (for that vector space), i.e.,

$$\mathbf{v} = \sum_{i=1}^{N} \operatorname{proj}_{\mathbf{e}^{(i)}}(\mathbf{v}) = \sum_{i=1}^{N} c_i \mathbf{e}^{(i)}$$

where, by definition, a unit vector has length unity and c_i is the projection of the vector $\mathbf{v}$ onto the unit vector $\mathbf{e}^{(i)}$, i.e.,

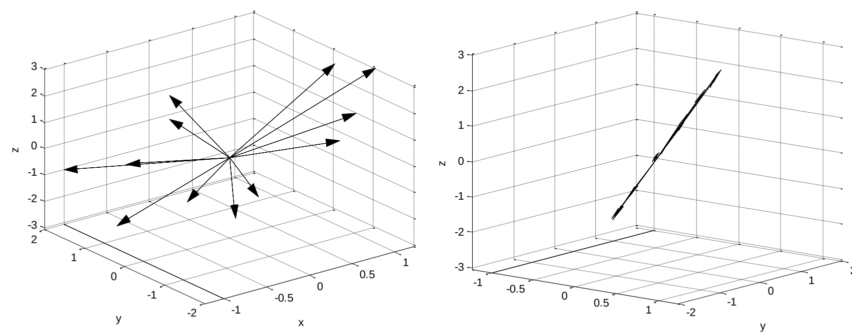
$$c_i = \left\langle \mathbf{v}, \mathbf{e}^{(i)} \right\rangle.$$

Example 6.25. In many applications, the standard choice for the orthonormal basis of a collection of (all possible real-valued) n-dimensional vectors is

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

6.26. Suppose we start with a collection of M n-dimensional vectors. Do these M vectors really need to be represented in n dimensions?

Example 6.27. Figure 15a shows a particular collection of 10 vectors in 3-D. When viewed from appropriate angle (as in Figure 15b), we can see that they all reside on a 2-D plane. We only need a two-vector (orthonormal) basis. All ten vectors can be represented as linear combinations of these two vectors.



(a) (b)

Figure 15: Ten vectors on a plane

Example 6.28. Consider the four vectors below:

$$\mathbf{v}^{(1)} = \begin{pmatrix} -2 \\ -6 \\ 2 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}.$$

They are all multiples of one another.

6.29. Similar idea applies to waveforms. In PAM, we have M waveforms that are simply multiples of a pulse $p(t)$. Therefore, one may represent them as points in one dimension as we had discussed in the previous subsection.

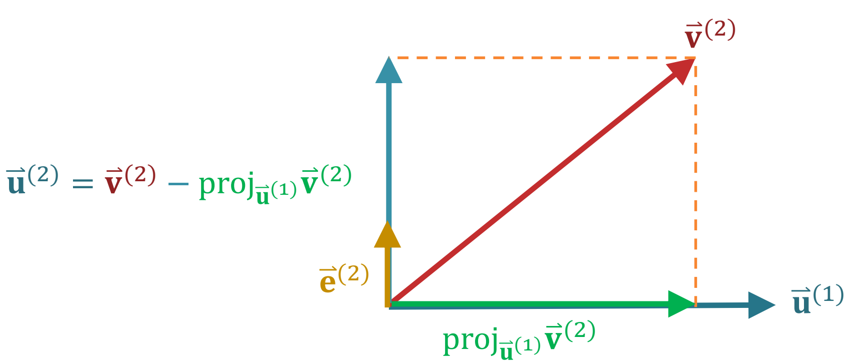
6.30. Gram-Schmidt Orthogonalization Procedure (GSOP) for constructing a collection of orthonormal vectors from a set of n-dimensional vectors $\mathbf{v}^{(i)}$, $1 \le i \le M$.

(a) Arbitrarily select a (nonzero) vector from the set, say, $\mathbf{v}^{(1)}$.
Let $\mathbf{u}^{(1)} = \mathbf{v}^{(1)}$.
Normalize its length to obtain the first vector: $\mathbf{e}^{(1)} = \frac{\mathbf{u}^{(1)}}{\left\|\mathbf{u}^{(1)}\right\|}$.

(b) Select an unselected vector from the set, say, $\mathbf{v}^{(2)}$.
Subtract the projection of $\mathbf{v}^{(2)}$ onto $\mathbf{u}^{(1)}$:

$$\mathbf{u}^{(2)} = \mathbf{v}^{(2)} - \operatorname{proj}_{\mathbf{u}^{(1)}}\left(\mathbf{v}^{(2)}\right) = \mathbf{v}^{(2)} - \frac{\left\langle \mathbf{v}^{(2)}, \mathbf{u}^{(1)} \right\rangle}{\left\langle \mathbf{u}^{(1)}, \mathbf{u}^{(1)} \right\rangle}\mathbf{u}^{(1)}$$
$$= \mathbf{v}^{(2)} - \left\langle \mathbf{v}^{(2)}, \mathbf{e}^{(1)} \right\rangle \mathbf{e}^{(1)}.$$

Then, we normalize the vector $\mathbf{u}^{(2)}$ to unit length: $\mathbf{e}^{(2)} = \frac{\mathbf{u}^{(2)}}{\left\|\mathbf{u}^{(2)}\right\|}$.

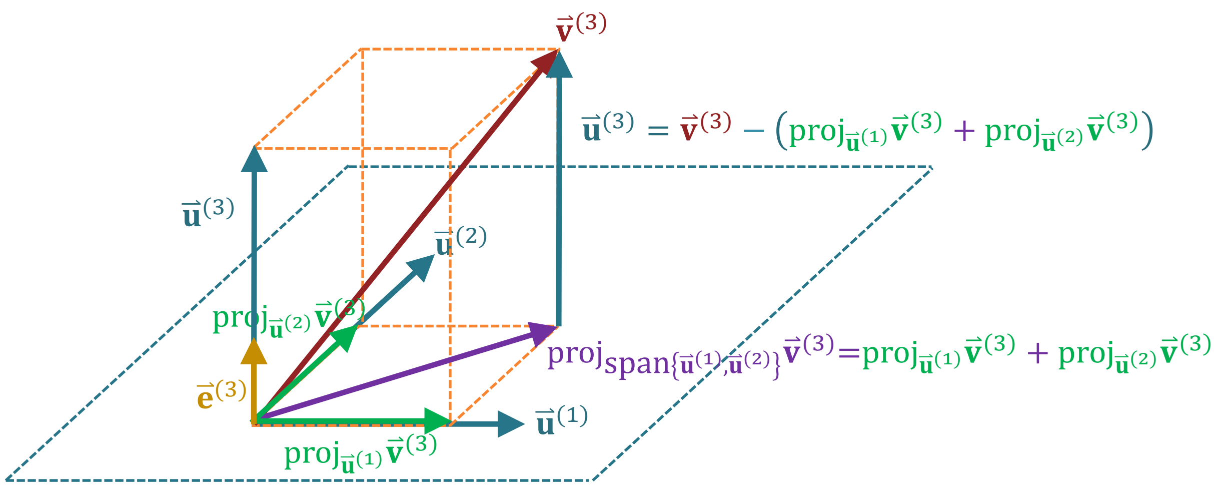


(c) Continue by selecting an unselected vector from the set, say, $\mathbf{v}^{(3)}$

and subtract the projections of $\mathbf{v}^{(3)}$ into $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$:

$$\begin{aligned}\mathbf{u}^{(3)} &= \mathbf{v}^{(3)} - \operatorname{proj}_{\mathbf{u}^{(1)}}\left(\mathbf{v}^{(3)}\right) - \operatorname{proj}_{\mathbf{u}^{(2)}}\left(\mathbf{v}^{(3)}\right) \\ &= \mathbf{v}^{(3)} - \frac{\left\langle \mathbf{v}^{(3)}, \mathbf{u}^{(1)}\right\rangle}{\left\langle \mathbf{u}^{(1)}, \mathbf{u}^{(1)}\right\rangle}\mathbf{u}^{(1)} - \frac{\left\langle \mathbf{v}^{(3)}, \mathbf{u}^{(2)}\right\rangle}{\left\langle \mathbf{u}^{(2)}, \mathbf{u}^{(2)}\right\rangle}\mathbf{u}^{(2)} \\ &= \mathbf{v}^{(3)} - \left\langle \mathbf{v}^{(3)}, \mathbf{e}^{(1)}\right\rangle \mathbf{e}^{(1)} - \left\langle \mathbf{v}^{(3)}, \mathbf{e}^{(2)}\right\rangle \mathbf{e}^{(2)}.\end{aligned}$$

Then, we normalize the vector $\mathbf{u}^{(3)}$ to unit length: $\mathbf{e}^{(3)} = \frac{\mathbf{u}^{(3)}}{\left\|\mathbf{u}^{(3)}\right\|}$.



(d) Continue this procedure for each of the remaining unselected vectors.

6.31. What do we get from GSOP?

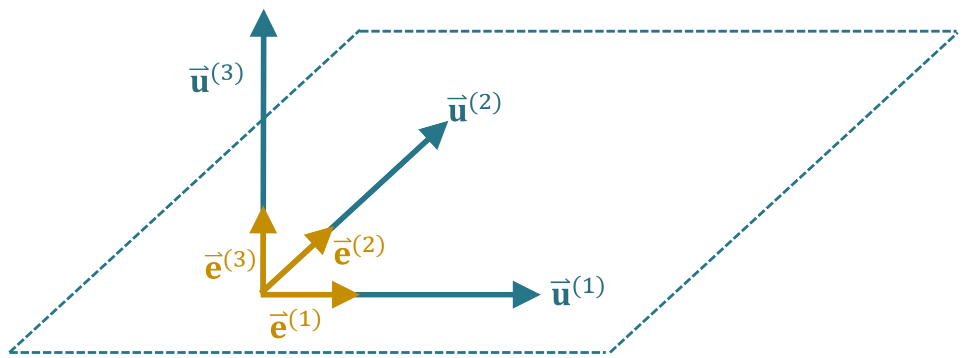
(a) A collection of N orthogonal vectors $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(N)}$ where

$$N \leq \min(M, n).$$

(i) We discard the zero $\mathbf{u}^{(k)}$ in the collection.

(ii) The $\mathbf{u}^{(k)}$ are re-indexed to replace the skipped values.

This is then normalized to be a collection of N orthonormal vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(N)}$.



(b) The collection $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\ldots$, $\mathbf{e}^{(N)}$ forms an orthonormal basis for the span of $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\ldots$, $\mathbf{v}^{(M)}$.

Similarly, the collection $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\ldots$, $\mathbf{u}^{(N)}$ forms an orthogonal basis for the span of $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\ldots$, $\mathbf{v}^{(M)}$.

(c) We can express $\mathbf{v}^{(j)}$ as

$$\mathbf{v}^{(j)} = \sum_{i=1}^{N} \mathrm{proj}_{\mathbf{e}^{(i)}}(\mathbf{v}^{(j)}) = \sum_{i=1}^{N} c_{i,j}\mathbf{e}^{(i)}$$

where $c_{i,j} = \left\langle \mathbf{v}^{(j)}, \mathbf{e}^{(i)} \right\rangle$. Then, the vector $\mathbf{c}^{(j)} = (c_{1,j}, c_{2,j}, \ldots, c_{N,j})^T$ gives the new coordinates of $\mathbf{v}^{(j)}$ based on the orthonormal basis from GSOP.

Example 6.32. Consider a collection of two vectors:

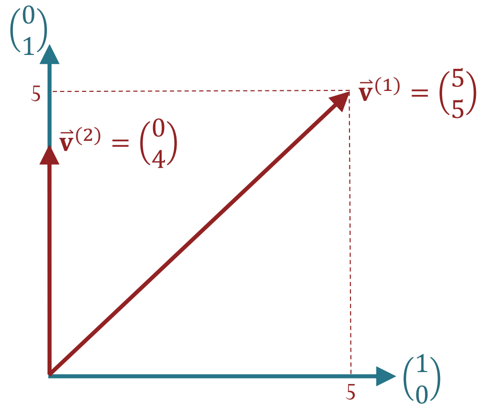
$$\mathbf{v}^{(1)} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{ and } \mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

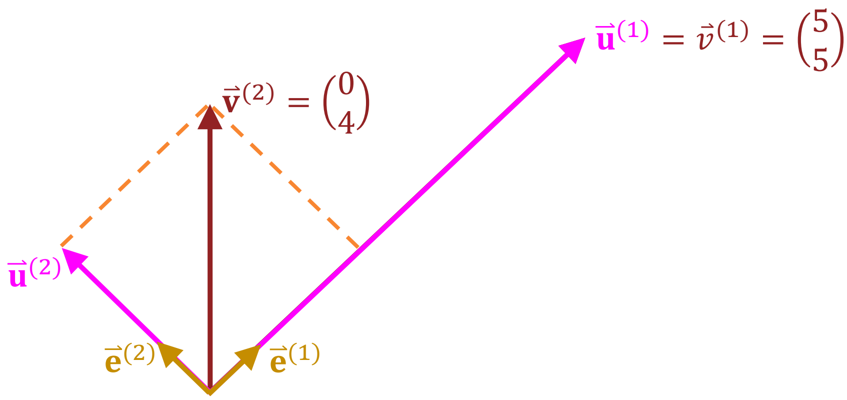
- In their original (default) coordinate systems, the basis contains two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- After applying the GSOP, we have two orthonormal vectors

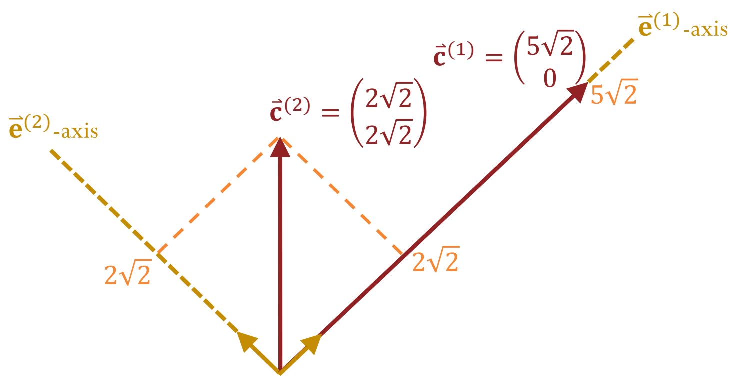
$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{e}^{(2)} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

- Using $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ as the new basis, we can express $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ in the new coordinate system as

$$\mathbf{c}^{(1)} = \begin{pmatrix} 5\sqrt{2} \\ 0 \end{pmatrix} \text{ and } \mathbf{e}^{(2)} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}.$$







6.33. Important properties: the transformation from $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(M)}$ to $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(M)}$ preserve many geometric quantities.

(a) Same inner product.

(b) Same norm.

(c) Same distance.